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MAXIMIZING THE MÖBIUS FUNCTION OF A POSET AND THE SUM OF THE BETTI NUMBERS OF THE ORDER COMPLEX

DMITRY N. KOZLOV

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This article concerns maximizing the Möbius function for different classes of partially ordered sets and the sum of the Betti numbers for their order complexes. First, we study how using various manipulations on posets can help limit the search range for the optimal poset. Then we find the sharp upper bound for the absolute value of the Möbius function on the class of posets of bounded width and classify the posets, which achieve this bound. Next, we consider the topological counterpart of the question. We find the sharp bound for the sum of the Betti numbers for the order complexes of arbitrary posets, posets of bounded width and ranked posets (with given rank function). We finish with a slight correction of the previous result of G. M. Ziegler.

1. Introduction

The Möbius function is a central tool in the analysis of partially ordered sets. Its absolute value unifies in a fascinating way the combinatorial complexity of a poset and the topological complexity of its order complex (the Möbius function of a poset is equal to the reduced Euler characteristic of its order complex). This makes it interesting to ask, *when does the absolute value of the Möbius function achieve its maximum*, when different conditions are imposed on the considered class of posets.

To illustrate a possible difficulty of a question of that type, let us mention the following open problem: *Given an n-element lattice L, what is the largest possible value of the absolute value of its Möbius function?* One can achieve $n^{2-\varepsilon}$ (for any $\varepsilon>0$ and sufficiently large n) by taking L to be the lattice of subspaces of a suitable finite dimensional vector space over a finite field (see [9, Example 3.10.2], [12, Section 5] or [10, Section 5]). It is conjectured that this is also best possible, i.e., $\lim_{n\to\infty}\mu_L(n)/n^2=0$.

G. M. Ziegler in [12] and, independently, E. E. Marenich in [7] solved the problem of maximizing the Möbius function of a poset with n elements. The so called *compression* technique that G. M. Ziegler used was inspired by the methods of extremal set theory, see [6]. After that some other interesting classes of posets have been considered in order to develop techniques for maximizing the Möbius function (see [12] for a quite thorough account).

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In this paper we consider the class of posets of bounded width, the techniques used in [12] seem to be inapplicable in this case. We settle this problem in Section 3, using so called *irreducible* elements. In Section 2 we define some properties of the classes of posets which guarantee the existence of posets maximizing the Möbius function and also having some special properties.

As mentioned above, the estimation of the Möbius function of a poset is equivalent to the estimation of the Euler characteristic of its order complex. On the other hand there are much finer invariants for topological complexity, namely the Betti numbers. The problem of maximizing the Euler characteristic and the sum of Betti numbers for an arbitrary simplicial complex with at most n vertices has been considered in [3, Theorem 1.4 and Appendix]. In Section 4 we consider this problem for the order complexes of arbitrary posets, ranked posets and posets with bounded width. It turns out that the compression technique mentioned previously has a topological analogue and can be used to find the maximum of the sum of the Betti numbers as well as of any particularly chosen Betti number.

Finally let us mention one interesting topic, where the maximization of the Möbius function comes up.

Lower bounds in computer science. The problem of estimating lower bounds is one of the hardest in modern computer science. One general method is to estimate the depth of a *decision tree*. An approach through the estimation of the proper Möbius function was first suggested in [5], later followed by [4]. One of the sample problems considered there is: Given n numbers decide whether at least k of them are equal. See the mentioned references for more details.

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2. Techniques

Let us briefly go through the terminology which we will use in this paper. We refer to [9, Chapter 3] and [1] for all poset notions not defined here (such as Möbius function, order ideal, dual order ideal etc.). Throughout this paper we only consider finite posets with adjoined maximum and minimum elements, which we denote $\hat{1}$ and $\hat{0}$. We denote $\bar{P}=P\setminus\{\hat{0},\hat{1}\}$ (note that $\bar{P}$ may still have maximum or minimum elements). $|P|$ denotes the number of elements in P. We say that x *covers* y if $x>y$ and there is no z such that $x>z>y$; we denote that by $x\to y$. We call $x\in P$ a *coatom* if $\hat{1}$ covers x. The set of coatoms of P is denoted by $\mathcal{C}(P)$. Recall that the *width* of a poset P is the number of elements in the maximal antichain (a set of incomparable elements) of P, usually denoted by $w(P)$. An element in P is called *irreducible* if it covers or is covered by exactly one element different from $\hat{0}$ and $\hat{1}$. If Q is an induced subposet of P, then we denote by $\mu_Q(\hat{0},\hat{1})$ or sometimes just μ_Q the Möbius function calculated over $\hat{0}\oplus(Q\cap\bar{P})\oplus\hat{1}$, where $\oplus$ denotes the ordinal sum. For a poset P and $x \in P$ we use the notation $P_{<x} = \{y \in P \,|\, y < x\}$, and, similarly $P_{\leq x}$, $P_{>x}$, $P_{\geq x}$.

We say that $C \subseteq P$ is a *chain* if any two elements of C are comparable. For a finite poset P one can construct its *order complex*, $\Delta(P)$, by taking the elements of P as vertices of the complex and the chains of P as simplices of $\Delta(P)$. When P has adjoined $\hat{0}$ and $\hat{1}$ we usually simply write $\Delta(P)$ instead of $\Delta(\bar{P})$. We denote the reduced ith Betti number of $\Delta(P)$ by $\beta_i(P)$ or just β_i. We also often write $H_k(P)$ instead of $H_k(\Delta(P))$. In general, any topological statement about posets concerns their order complexes. Throughout the paper the homology is taken with coefficients in a field. All other terminology from algebraic topology that we use (such as link (lk) and star (st) constructions, contractibility, etc.) can be found in [8].

By K we usually denote some class of posets, having some explicitly stated property. Let us specify a few such classes.

Definition 2.1. We say that a class K is *closed under deletion of elements* if any induced subposet of a poset in K also lies in K. In other words deletion of an element (other than $\hat{0}$ and $\hat{1}$) from some $P \in K$ and inheriting the partial order of P results in a poset $Q \in K$.

Definition 2.2. We say that a class K is *closed under deletion of relations* if for any poset Q and poset $P \in K$, such that P and Q are defined on the same set of elements and the partial order on Q satisfies: if $x \leq y$ in Q then $x \leq y$ in P, we get that $Q \in K$. In other words the identity induces an order-preserving map $Q \to P$.

The general problem considered here is: *given some class K of posets of given size, what is the maximal absolute value of Möbius function on K?* If one is interested in topological complexity of a poset one can also ask: what is the maximal value of the sum of the (reduced) Betti numbers? Of course, it is also interesting to classify the posets for which the maximum is achieved. We call such posets optimal (or sometimes extremal) with respect to Möbius function or to the sum of the Betti numbers.

In the following we show how properties of the considered class can help us limit the range of search for the optimal poset.

Definition 2.3. We call a poset P *sign-matching* if for any $x \in P$, $\mu_P(\hat{0},\hat{1})$ and $\mu_P(\hat{0},x) \cdot \mu_P(x,\hat{1})$ are both different from zero and have the same sign.

Proposition 2.4. *Let K be class of posets closed under deletion of elements. Assume that not all posets in K have Möbius function equal to 0. Then there exist posets $P_1, P_2 \in K$ such that*

(1) *P_1 is sign-matching and optimal with respect to the Möbius function;*
(2) *P_2 has no elements x, such that either $\Delta(P_{<x})$ or $\Delta(P_{>x})$ is contractible (so called generalized irreducibles) and is optimal with respect to the sum of the Betti numbers.*

In both cases, if K is closed under deletion of irreducibles, then there exists an optimal poset without irreducibles.

Proof. (1) Let P be an optimal poset in K with respect to the Möbius function. If $\mu_P(\hat{0},\hat{1}) = 0$ then all posets in K have the Möbius function equal to 0, which is a contradiction. Hence, if P is not sign-matching, then for some $x \in P$ either $\mu_P(\hat{0},x)\cdot\mu_P(x,\hat{1})$ and $\mu_P(\hat{0},\hat{1})$ have different signs or $\mu_P(\hat{0},x)\cdot\mu_P(x,\hat{1})=0$. In both cases it follows from

$$\mu_{P\setminus\{x\}}(\hat{0},\hat{1}) = -\mu_P(\hat{0},x)\mu_P(x,\hat{1}) + \mu_P(\hat{0},\hat{1})$$

that

$$|\mu_{P\setminus\{x\}}(\hat{0},\hat{1})| \geq |\mu_P(\hat{0},\hat{1})|,$$

which means that $P\setminus\{x\}$ is optimal as well. Iterating this process we will eventually obtain a sign-matching optimal poset.

(2) follows by analogous argument from [11, Proposition 6.1], which states that the inclusion of $\Delta(P\setminus\{x\})$ into $\Delta(P)$ induces a homotopy equivalence. ∎

The following lemma (proof of which is left to the reader) often allows us to proceed by induction on the number of coatoms.

Lemma 2.5. *Let P be a poset with t coatoms and without irreducible elements. Let x be one of coatoms, then $P\setminus\{x\}$ has exactly $t-1$ coatoms.*

Let us see now how far we can get by deleting relations.

Definition 2.6. A poset P has *sign alternating Möbius function*, if for any $x \in P$ and any maximal chain in $P_{\leq x}$ of length l, we have $(-1)^l\mu_P(\hat{0},x)>0$. In particular $\mu_P(\hat{0},x)\neq 0$.

Definition 2.7. P is a *locally optimal* poset (with respect to the Möbius function) if for all x:

$$|\mu_P(\hat{0},x)\cdot\mu_P(x,\hat{1})| = \max|\mu_I\cdot\mu_J|,$$

where the maximum is taken over all I and J, such that I is an order ideal of $P_{<x}$, J is a dual order ideal of $P_{>x}$, and $\mu_I\cdot\mu_J$ has the same sign as $\mu_P(\hat{0},x)\cdot\mu_P(x,\hat{1})$, here $\mu_I=\mu(I\cup\{\hat{1}\})$, $\mu_J=\mu(J\cup\{\hat{0}\})$. See Figure 1 for an illustration.

In other words, the Möbius function of a locally optimal poset cannot be improved by changing the partial ordering locally on one element.

Lemma 2.8. *Let K be a class of posets closed under the deletion of elements and relations, then there exists an optimal (with respect to the Möbius function) poset $P\in K$, which is also locally optimal.*

Proof. By Proposition 2.4 we can find an optimal poset P, which is also sign-matching. Assume P is not locally optimal and take I the order ideal of $P_{<x}$ and J

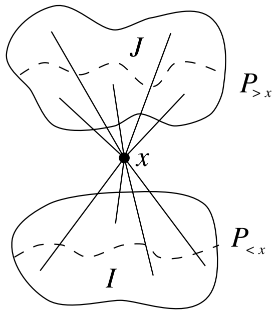

Fig. 1

the dual order ideal of $P_{>x}$, such that $\mu_I \cdot \mu_J$ has the same sign as $\mu_P(\hat{0},x) \cdot \mu_P(x,\hat{1})$ and $|\mu_I \cdot \mu_J| > |\mu_P(\hat{0},x) \cdot \mu_P(x,\hat{1})|$. Consider a weak subposet Q with the same set of elements as P and relations given by :

(1) for $y \neq x$, $z \neq x$, $y < z$ in Q iff $y < z$ in P;

(2) $x < z$ in Q iff $z \in J$;

(3) $y < x$ in Q iff $y \in I$.

These relations give a partial order because of the choice of I and J. On the other hand

$$|\mu_Q(\hat{0},\hat{1})| = |\mu_{Q\setminus\{x\}}(\hat{0},\hat{1}) + \mu_Q(\hat{0},x)\mu_Q(x,\hat{1})| = |\mu_{P\setminus\{x\}}(\hat{0},\hat{1}) + \mu_I\mu_J|$$

$$= |\mu_P(\hat{0},\hat{1}) - \mu_P(\hat{0},x)\mu_P(x,\hat{1}) + \mu_I\mu_J| > |\mu_P(\hat{0},\hat{1})|. \tag{2.1}$$

The last inequality follows from the fact that P is sign-matching and the choice of I and J. ∎

Corollary 2.9. *Let K be as in Lemma 2.8, then there exists an optimal (with respect to the Möbius function) poset $P \in K$, with sign alternating Möbius function.*

Proof. It is enough to prove that if x covers y then

$$\mu_P(\hat{0},x)\mu_P(\hat{0},y) < 0. \tag{2.2}$$

By Lemma 2.8 we can take an optimal poset P, which is locally optimal. Assume that there exist x and y, such that x covers y and (2.2) is false. Assume for simplicity that both $\mu_P(\hat{0},x)$ and $\mu_P(\hat{0},y)$ are positive (the other case can be treated analogously). Now take $I = P_{<x} \setminus \{y\}$, $J = P_{>x}$. Then $\mu_I = \mu_P(\hat{0},x) + \mu_P(\hat{0},y) > 0$ and $\mu_J = \mu_P(x,\hat{1})$, hence $\mu_I\mu_J$ and $\mu_P(\hat{0},x)\mu_P(x,\hat{1})$ have the same sign. One sees also that $|\mu_I\mu_J| > |\mu_P(\hat{0},x)\mu_P(x,\hat{1})|$, which contradicts to the fact that P is locally optimal. ∎

To illustrate the result above, we give a definition.

Definition 2.10. We call P a *poset with bounded comparability* if there exists a number m such that each element of P is comparable with at most m elements. Sometimes we just say that P is *m-comparable*.

One can easily see that the class of all m-comparable posets with at most n elements is closed under deletion of elements and relations, hence we can conclude from Corollary 2.9 that there is an optimal poset among those with sign alternating Möbius function and also there exists an optimal poset among locally optimal ones.

3. Maximizing the Möbius function of posets with bounded width

Definition 3.1. Let n and w be positive integers and define

$$\mu_P(n,w) = \max_Q |\mu_Q(\hat{0},\hat{1})|,$$

where the maximum is taken over all posets Q such that $|Q| = n$ and $w(Q) = w$.

Define also

$$\mu_P(n) = \max_{n \geq w \geq 1} \mu_P(n,w).$$

Let $K(n,w)$ denote the class of posets of width at most w and with at most n elements. $K(n,w)$ is obviously closed under deletion of elements, so some results in the previous section can be applied.

Definition 3.2. We say that a poset P is of *level type* $(p_1,\dots,p_r)$ if $P \simeq \hat{0} \oplus p_1 1 \oplus \dots \oplus p_r 1 \oplus \hat{1}$, where $\oplus$ denotes the ordinal sum, and $k1$ denotes an antichain consisting of k elements. For brevity we will often just say that a poset is of level type.

It has been shown in [12, Theorem 2.5] that

$$\mu_P(n) = \max_{r \geq 1} \max_{\substack{p_1+\dots+p_r=n \\ p_i \geq 1}} \prod_{i=1}^{r} (p_i - 1),$$

and that the extremal posets are of level type. So, one could guess that limiting the width would only result in the condition $w \geq p_i$ and that the extremal posets would be of level type. This is partially true.

Main Theorem 3.3.

$$\mu_P(n,w) = \max_{r \geq 1} \max_{\substack{p_1+\dots+p_r \leq n \\ w \geq p_i \geq 1}} \prod_{i=1}^{r} (p_i - 1).$$

If P is an extremal poset then there exists unique $l_{n,w}$ and $x_1,\dots,x_{l_{n,w}}$ elements of P, such that for all i, x_i is irreducible in $P \setminus \{x_1,\dots,x_{i-1}\}$ and $P \setminus \{x_1,\dots,x_{l_{n,w}}\}$

is of level type. More specifically, $l_{n,1}=n-1$, $l_{n,2}\le n-2$, $l_{n,3}\le 2$, $l_{n,4}\le 1$ *and* $l_{n,w}=0$ *for* $w\ge 5$.

This theorem provides us with an alternative proof of the general result [12, Theorem 2.5]. But first we need a lemma.

Lemma 3.4. *The function* $\mu_P(n,w)$ *is monotone (not necessarily strictly) in both variables, i.e.,*
(1) $\mu_P(n,w+1)\ge\mu_P(n,w)$;
(2) $\mu_P(n+1,w)\ge\mu_P(n,w)$.

Proof. (1) is obvious. To prove (2) consider P, a poset of width w with n elements. We can always add x to P and obtain a new poset $Q\in K$ so that x is irreducible in Q and $\Delta(Q\setminus\{x\})\simeq\Delta(P)$. ∎

Lemma 3.5. *Let P be a poset with at most n elements, m coatoms, length $l+1$ and with $w(P)\le w$. Then*

$$|\mu_P(\hat{0},\hat{1})|\le(m-1)\max_{l-1\ge k\ge 1}\max_{\substack{p_1+\ldots+p_k\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{k}(p_i-1).$$

Proof. Note that the deletion of irreducible elements preserves the number of coatoms. Hence, by the Proposition 2.4, we can assume without loss of generality that P has no irreducible elements. Let $x\in\mathcal{C}(P)$. Consider $P_1=P_{<x}$, $P_2=P\setminus\{x\}$. By Lemma 2.5, P_2 has exactly $m-1$ coatoms. Observe also that P_1 has the length at most l and $|P_1|\le n-m$. Hence

$$|\mu_P(\hat{0},\hat{1})|=|-\mu_P(\hat{0},x)+\mu_{P\setminus\{x\}}(\hat{0},\hat{1})|=|-\mu_{P_1}+\mu_{P_2}|\le|\mu_{P_1}|+|\mu_{P_2}|$$

$$\le\max_{l-1\ge k\ge 1}\max_{\substack{p_1+\ldots+p_k\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{k}(p_i-1)+(m-2)\max_{l-1\ge k\ge 1}\max_{\substack{p_1+\ldots+p_k\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{k}(p_i-1)$$

$$=(m-1)\max_{l-1\ge k\ge 1}\max_{\substack{p_1+\ldots+p_k\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{k}(p_i-1). \tag{3.1}$$ ∎

The bound in Theorem 3.3 follows immediately from Lemma 3.5. It is clear that the bound is achieved by proper posets of level type. It turns out that for small w there are more posets achieving maximum.

Classification

Table 1 displays numerical values for $\mu_P(n)$. Although this has been done in [12, page 211], we have corrected the case $n=5k+1$.

The following is a case-by-case consideration. Rather than ordering cases according to increasing values of w, we order them by increasing difficulty.

(1) $\underline{w=1}$. Any poset of this width is a chain.

(2) $\underline{w\geq 6}$. As we see from Table 1, all the extremal posets have width at most 6 except for $n=7$, where the extremal posets have widths 4 resp. 7. So with only one exception the extremal posets for $w=6$ coincide with those in the general case.

(3) $\underline{w=5}$. The only nontrivial case $n=11$ is left to the reader.

(4) $\underline{w=2}$. In this case we have $\mu_P(n,2)=1$ for $n\geq 2$. Again this is left as an exercise.

(5) $\underline{w=4}$. In this case the answer is given by the Table 2.

n	the best partition(s)	$\mu_P(n)$
$1,\ldots,6$	n	$n-1$
7	$3+4$, 7	$6=2\cdot 3$
11	$5+6$	$4\cdot 5$
$5k$, $k>0$	$5+5+\ldots+5$	4^k
$5k+1$, $k>2$	$4+4+4+4+5+\ldots+5$	$3^4\cdot 4^{k-3}$
$5k+2$, $k>1$	$4+4+4+5+\ldots+5$	$3^3\cdot 4^{k-2}$
$5k+3$, $k>0$	$4+4+5+\ldots+5$	$3^2\cdot 4^{k-1}$
$5k+4$, $k>0$	$4+5+\ldots+5$	$3\cdot 4^k$

Table 1

n	the best partition(s)	$\mu_P(n,4)$
$1,2,3$	n	$n-1$
$4k$, $k>0$	$4+\ldots+4$	3^k
$4k+1$, $k>0$	$4+\ldots+4$ (+1 irr.)	3^k
$4k+2$, $k>0$	$3+3+4+\ldots+4$	$2^2\cdot 3^{k-1}$
$4k+3$, $k>0$	$3+4+\ldots+4$	$2\cdot 3^k$

Table 2

a) $n=4k$. The extremal poset cannot have any irreducible elements, since $\mu_P(4k-1,4)<\mu_P(4k,4)$. According to the Lemma 3.5, the number of coatoms has to be 4. Furthermore, we must have the equality throughout (3.1), this implies that if x is a coatom, then $P_{\geq x}$ is a poset with $4k-4$ elements (as usually x and $\hat{0}$ are not counted), with maximal Möbius function. By induction we can assume that $P_{\geq x}$ is of level type $(4,4,\ldots,4)$. This is true for any coatom of P, hence $P=\{\mathcal{C}(P)\}\oplus(P\setminus\{\mathcal{C}(P)\})$ and the result follows by induction.

Cases $n=4k+2$ and $n=4k+3$ are very similar. Their consideration is left to the reader.

b) $n = 4k+1$. This case is principally different from those above, as the extremal poset P can have one irreducible element. If it does, then we just delete this element and end up with the already considered case of the poset with $4k$ elements.

Our task now is to prove that P must have an irreducible element. This is obvious for $n=5$. Let us proceed by induction. According to the Lemma 3.5 again we know that P has exactly four coatoms. If each coatom of P is larger than all the elements in $P \setminus \{\mathcal{C}(P)\}$, then one can write $P = \{\mathcal{C}(P)\} \oplus (P \setminus \{\mathcal{C}(P)\})$. The poset $P \setminus \{\mathcal{C}(P)\}$ has $4(k-1)+1$ elements and is extremal, hence by induction it must have an irreducible element, since any irreducible element of $P \setminus \{\mathcal{C}(P)\}$ is also irreducible in P, we get the result. So there must exist some $x \in \mathcal{C}(P)$ and some $a \in \mathcal{C}(P \setminus \{\mathcal{C}(P)\})$, such that $P_{<x} = P \setminus \{\mathcal{C}(P), a\}$. $P_{<x}$ has $4(k-1)$ elements and is extremal, hence it is of level type $(4, 4, \dots, 4)$. Now a must be comparable to some of the four elements in $\mathcal{C}(P_{<x})$ (since $w(P) < 5$). On the other hand x and a are incomparable, so a must actually cover some element from $\mathcal{C}(P_{<x})$, this element in its turn is larger than all the elements in $P_{<x} \setminus \{\mathcal{C}(P_{<x}\})$. Hence elements in $\mathcal{C}(P_{<x})$ are the only elements of P, which a possibly can cover. Furthermore a is not irreducible, so it must cover at least 2 of them. An easy calculation of the Möbius function shows that $\mu_P(\hat{0}, a)$ has a sign $(-1)^k$, while $\mu_P(a, \hat{1})$ is positive (because coatoms of P are the only elements, which can cover a). One also sees that $\mu_P(\hat{0}, \hat{1})$ must have a sign $(-1)^{k+1}$, since $\mu_P(\hat{0}, x)$ have a sign $(-1)^k$, and we have to have equality in (3.1). Thus $\mu_P(\hat{0}, a)\mu_P(a, \hat{1})$ and $\mu_P(\hat{0}, \hat{1})$ have different signs, which contradicts to the Corollary 2.9.

(6) $\underline{w=3}$. Also here we give the answer in the table form: Table 3.

a) $n = 3k$. The reasoning here is completely similar to the one for the case $w=4$, $n=4k$.

b) $n = 3k+1$. This case is essentially the same as for $w=4$, $n=4k+1$.

c) $n = 3k+2$. Let P be an extremal poset. If P has an irreducible element, then we delete it and end up in the case $n=3k+1$. Otherwise, if P does not have any irreducibles, by Lemma 3.5 it has either 2 or 3 coatoms. The 2 coatoms case requires examining the extremal posets for $P_{<x}$ and $P_{<y}$, where x and y are the 2 coatoms. The argument for the 3 coatoms case is similar to that of the width 4, $n=4k+1$ case, where instead one considers the two situations $|P_{<x}| = 3(k-1)$ and $|P_{<x}| = 3(k-1)+1$.

n	the best partition(s)	$\mu_P(n,3)$
$1,2$	n	$n-1$
$3k$, $k>0$	$3+\ldots+3$	2^k
$3k+1$, $k>0$	$3+\ldots+3$ (+1 irr.)	2^k
$3k+2$, $k>0$	$3+\ldots+3$ (+2 irr.) or $2+3+\ldots+3$	2^k

Table 3

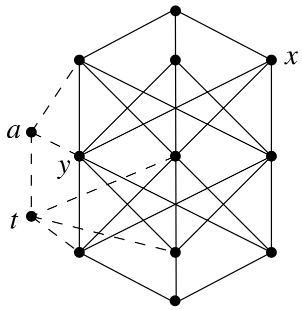

Fig. 2

The Figure 2 shows what the typical poset with $n=3k+2$ (here $k=3$) and $w=3$ can look like.

4. Maximizing the sum of the Betti numbers

The Möbius function of a poset is equal to the reduced Euler characteristic of its order complex. So, when estimating it, we may think that we estimate the topological complexity of our poset. On the other hand, the Euler characteristic is an alternating sum of the Betti numbers. Hence it would be interesting to estimate the sum of the Betti numbers as well. Let us begin with computing the Betti numbers of the level type posets.

Proposition 4.1. *Let P be a poset of level type $(p_1,\dots,p_k)$. Then*

$$\beta_{-1}(P)=\beta_0(P)=\ldots=\beta_{k-2}(P)=0,\ \beta_{k-1}(P)=\prod_{i=1}^{k}(p_i-1).$$

Proof. This is a well-known fact which can be proved by induction using Mayer-Vietoris sequence. Alternatively one can view the level type poset as a matroid and use [2, Theorems 7.3.3, 7.7.2]. ∎

It turns out that the sharp bound for the sum of the Betti numbers is the same as for the Möbius function.

Theorem 4.2. *Let P be a poset of length $l+1$ with at most n elements and with width at most w, then*

$$\sum_{i=0}^{l-1}\beta_i(P)\le\max_{l\ge r\ge 1}\ \max_{\substack{p_1+\ldots+p_r\le n\\ w\ge p_i\ge 1}}\ \prod_{i=1}^{r}(p_i-1),$$

and the maximum is achieved exactly when $\sum_{i=0}^{l-1}\beta_i(P) = |\mu_P(\hat{0},\hat{1})|$, *i.e., when there is homology only in even or only in odd dimensions. The extremal posets are exactly the same as in the previous section. Moreover, for any* $m \le l-1$

$$\beta_m \le \max_{\substack{p_1+\ldots+p_{m+1}\le n\\ w\ge p_i\ge 1}} \prod_{i=1}^{m+1}(p_i-1).$$

Exactly as in the previous section we prove the crucial lemma. The method of the proof is similar to that of Lemma 3.5.

Lemma 4.3. *Let* P *be a poset of length* $l+1$ *with at most* n *elements, width at most* w *and with* m *coatoms, then*

$$\sum_{i=0}^{l-1}\beta_i(P) \le (m-1)\max_{l-1\ge r\ge 1}\max_{\substack{p_1+\ldots+p_r\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{r}(p_i-1).$$

Moreover, for any k

$$\beta_k(P) \le (m-1)\max_{\substack{p_1+\ldots+p_k\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{k}(p_i-1).$$

Proof. By Proposition 2.4 we can assume that P does not contain any irreducible elements. Let $x\in\mathcal{C}(P)$. Let $P_1 = P_{\le x}$, $P_2 = P\setminus\{x\}$ and set $A = P_1\cap P_2$. Clearly, $\Delta(P_1)$ is a cone, and we have

$$\Delta(P) = \Delta(P_1)\cup\Delta(P_2),\ \Delta(A) = \Delta(P_1)\cap\Delta(P_2).$$

If $A=\emptyset$ then it is easy to see that x is not comparable to any element except for $\hat{0}$ and $\hat{1}$, and the statement follows easily by induction. So assume $A\neq\emptyset$. A has length at most l, hence $H_{l-1}(A)=0$.

Consider the Mayer-Vietoris sequence :

$$0\to H_{l-1}(P_2)\longrightarrow H_{l-1}(P)\longrightarrow H_{l-2}(A)\longrightarrow\ldots\longrightarrow H_0(P_2)\longrightarrow H_0(P)\to 0.$$

This is a long exact sequence, hence we obtain the l inequalities

$$\beta_{l-1}(P)\le\beta_{l-1}(P_2)+\beta_{l-2}(A),\ \ldots\ ,\beta_0(P)\le\beta_0(P_2).$$

Summing up gives

$$\sum_{i=0}^{l-1}\beta_i(P)\le\sum_{i=0}^{l-1}\beta_i(P_2)+\sum_{i=0}^{l-2}\beta_i(A)$$

$$\le (m-2)\max_{l-1\ge r\ge 1}\max_{\substack{p_1+\ldots+p_r\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{r}(p_i-1)+\max_{l-1\ge r\ge 1}\max_{\substack{p_1+\ldots+p_r\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{r}(p_i-1)$$

$$= (m-1)\max_{l-1\ge r\ge 1}\max_{\substack{p_1+\ldots+p_r\le n-m\\ w\ge p_i\ge 1}}\prod_{i=1}^{r}(p_i-1). \tag{4.1}$$

To prove the second part observe that:

$$\beta_k(P) \le \beta_k(P_2) + \beta_{k-1}(A)$$

$$\le (m-2) \max_{\substack{p_1+\ldots+p_k \le n-m \\ w \ge p_i \ge 1}} \prod_{i=1}^{k}(p_i - 1) + \max_{\substack{p_1+\ldots+p_k \le n-m \\ w \ge p_i \ge 1}} \prod_{i=1}^{k}(p_i - 1)$$

$$= (m-1) \max_{\substack{p_1+\ldots+p_k \le n-m \\ w \ge p_i \ge 1}} \prod_{i=1}^{k}(p_i - 1). \tag{4.2}$$

This proves all the bounds in Theorem 4.2.

To understand which posets are optimal we review the proof and see when all the inequalities are actually equalities. Thus the conditions imposed are essentially the same as in the previous section, so the classification there can be applied again without any major changes. ∎

Corollary 4.4. *Let P be a poset with n elements, then*

$$\sum \beta_i(P) \le \max_{r \ge 1} \max_{\substack{p_1+\ldots+p_r = n \\ p_i \ge 1}} \prod_{i=1}^{r}(p_i - 1)$$

and the bound is achieved by the corresponding level type posets only.

Moreover, for any m

$$\beta_m(P) \le \max_{p_1+\ldots+p_{m+1} = n} \prod_{i=1}^{m+1}(p_i - 1).$$

Proof. Both statements follow from Theorem 4.2 by putting $w = n$. However we shall give a different proof in order to illustrate our generalization of Ziegler's compression technique to the case of simplicial complexes.

Let x, y be two coatoms of P, such that

$$\sum \beta_i(\hat{0}, x) \ge \sum \beta_i(\hat{0}, y).$$

Let Q be a poset on the same set as P and with partial ordering given by:

(1) $z \le_Q t \Leftrightarrow z \le_P t$, $t \ne y$;

(2) $z \le_Q y \Leftrightarrow z \le_P x$.

We call Q a *compression* of P. Set

$$P_1 = P_{\le y},\ P_2 = P \setminus \{y\},\ A_1 = P_1 \cap P_2 = P_{<y},$$

$$Q_1 = Q_{\le y},\ Q_2 = Q \setminus \{y\} = P_2,\ A_2 = Q_1 \cap Q_2 = Q_{<y} = Q_{<x}.$$

$\Delta(P_1)$ and $\Delta(Q_1)$ are cones. Using a Mayer-Vietoris sequence as before we obtain

$$\sum \beta_i(P) \leq \sum \beta_i(P_2) + \sum \beta_i(A_1).$$

When $P = Q$ we get equality. The reason for that is that for all k the map i_* in $H_k(A_2) \xrightarrow{i_*} H_k(Q_1) \oplus H_k(Q_2)$ is a zero-map, as $\Delta(Q_1)$ is a cone and any cycle in $\Delta(A_2)$ is mapped to the cycle inside the cone $\Delta(Q_{\leq x})$ in $\Delta(Q_2)$. Hence

$$\begin{aligned} \sum \beta_i(Q) &= \sum \beta_i(Q_2) + \sum \beta_i(A_2) = \sum \beta_i(P_2) + \sum \beta_i(\hat{0}, x) \\ &\geq \sum \beta_i(P_2) + \sum \beta_i(\hat{0}, y) \geq \sum \beta_i(P). \end{aligned} \tag{4.3}$$

So the compressed poset has larger or equal sum of Betti numbers. We continue compression until it stops, and denote the final poset by P again. Let $T = \mathcal{C}(P)$ and $t = |T|$. Then $t-1$ iterations of the first equality in (4.3) gives

$$\sum \beta_i(P) = (t-1) \sum \beta_i(P \setminus T)$$

and the result follows by induction.

The case when the sum of the Betti numbers is replaced by a single β_m can be treated along the same lines. ∎

Let us say just a few words about generalization of the compression technique to arbitrary simplicial complexes. The operation used in the proof of Corollary 4.4 can be generalized as follows: let Δ be an arbitrary simplicial complex, $x, y \in \Delta$, such that $x \notin \operatorname{lk} y$ and $y \notin \operatorname{lk} x$, and assume that

$$\sum \beta_i(\operatorname{lk} x) \geq \sum \beta_i(\operatorname{lk} y).$$

Take away y and adjoin an exact copy of x. The obtained complex Δ' will have larger or equal sum of Betti numbers, which can be proved using the Mayer-Vietoris sequence in the same way as we did above, if we take

$$P_1 = \operatorname{st} y,\ P_2 = \Delta \setminus \{F \mid y \in F\},\ A_1 = P_1 \cap P_2 = \operatorname{lk} y$$

and appropriate Q_1 and Q_2. One can also maximize any particularly chosen Betti number using the same method.

Next we prove the translation of [12, Theorem 3.2] into topological language. For a ranked poset P and $x \in P$, let $r(x)$ denote the rank of x.

Theorem 4.5. *Let P be a bounded poset of rank $l+1$, such that $P \setminus \hat{1}$ is ranked with rank generating function $1 + \sum_{i=1}^{l} p_i t^i$. Let s be the minimal number i such that $p_i = 1$ (if such p_i does not exist we put $s = l+1$), then*

$$\sum \beta_i(P) \leq \prod_{i=1}^{s-1} (p_i - 1). \tag{4.4}$$

This bound is sharp, since it is achieved by the level type posets.

Note. The theorem covers in particular the case of graded posets of some given length.

Proof. The proof is by induction on the number of elements of $\bar{P}$. The base case $|\bar{P}|=0$ is immediate, since both sides of (4.4) are equal to 1. Let us proceed with the proof of the induction step.

If for every coatom x we have $r(x)\geq s$, then $\Delta(P)$ is a cone with apex a (where a is the unique vertex satisfying $r(a)=s$), and hence the left hand side of (4.4) is equal to 0, while the right hand side is always positive.

If this is not the case, let x be a coatom of P such that no other coatom has smaller rank. Clearly, $r(x)<s$. We split the rest of the proof into two cases.

Case 1. $p_{r(x)} \geq 3$. Let $j=r(x)$. Set $P_1=P_{\leq x}$, $P_2=P\setminus\{x\}$ and $A=P_1\cap P_2$. Clearly $\Delta(P)=\Delta(P_1)\cup\Delta(P_2)$, $\Delta(A)=\Delta(P_1)\cap\Delta(P_2)$ and $\Delta(P_1)$ is a cone. Using Mayer-Vietoris sequence in the same way as in Lemma 4.3, we obtain

$$\sum\beta_i(P)\leq\sum\beta_i(P_2)+\sum\beta_i(A)\leq(p_j-2)\prod_{\substack{i=1\\i\neq j}}^{s-1}(p_i-1)+\prod_{i=1}^{j'-1}(p'_i-1)$$

$$\leq(p_j-2)\prod_{\substack{i=1\\i\neq j}}^{s-1}(p_i-1)+\prod_{\substack{i=1\\i\neq j}}^{s-1}(p_i-1)=\prod_{i=1}^{s-1}(p_i-1), \tag{4.5}$$

where $j'\leq j$ and $p'_i\leq p_i$ for $1\leq i\leq j'-1$.

Case 2. $p_{r(x)}=2$. Let y be the unique element satisfying $r(x)=r(y)$, $y\neq x$. Set $P_1=P_{\leq x}$, $P_2=P_{\leq y}\cup P_{\geq y}$ and $A=P_1\cap P_2$. Every element of P is comparable to at least one of the elements x and y, hence $\Delta(P)=\Delta(P_1)\cup\Delta(P_2)$. Since both $\Delta(P_1)$ and $\Delta(P_2)$ are cones, we conclude from the Mayer-Vietoris sequence that

$$\sum\beta_i(P)=\sum\beta_i(A)\leq\prod_{i=1}^{s'-1}(p'_i-1)\leq\prod_{i=1}^{s-1}(p_i-1),$$

where $s'\leq r(x)<s$ and $p'_i\leq p_i$ for $1\leq i\leq s'-1$. ∎

5. Remark

In this section we make a minor correction to the formulation and the proof of [12, Theorem 3.2]. In order to read it, the reader needs to view it in combination with [12]. Figure 3 shows one example of a poset which is missing in the classification in [Theorem 3.2][12].

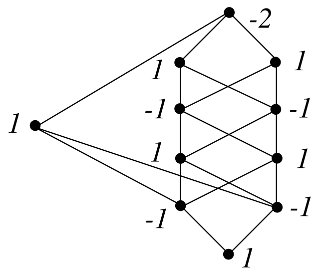

Fig. 3

The point in the proof which has to be corrected is that the equality on p. 213, next after the inequality (3.4), namely

$$\prod_{i=r(x)+1}^{s-1} (p_i - 1) = 1$$

does not imply that $r(x) = s-1$, but only that

$$p_{r(x)+1}\, , \, p_{r(x)+2}\, , \, \dots \, , \, p_{s-1} = 2.$$

The rest of the proof goes through. Thus to cover also this case we have to change part (ii) of Theorem 3.2 in [12] into:

the elements in $P \backslash P_{[t']}$ do not change the absolute value of the Möbius function of P, that is either $\mu_P(\hat{0}, x) = 0$ or we have a rank level $r(x)$ consisting of two elements x and y, such that $\mu_P(\hat{0}, x) = \mu_P(\hat{0}, y) = \mu_{P_{[r(x)-1]}}(\hat{0}, \hat{1})$, (this only changes the sign of the Möbius function, not its absolute value).

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Dmitry N. Kozlov

Department of Mathematics,
Royal Institute of Technology,
S-100 44, Stockholm, Sweden
kozlov@math.kth.se